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To the question of vector analysis of the uniform rotation of a material point around a fixed axis

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Abstract. *On the basis of the conducted vector analysis of the process of uniform rotation of a material point around a fixed axis it was established that the generally accepted kinematic parameters of the rotation of a material point in the form of tangential velocity and centripetal acceleration are necessary, but insufficient for the complete characterization of the kinematics of the process of rotation of a material point around a fixed axis. The centrifugal acceleration, which is equal in magnitude and opposite in direction to centripetal acceleration; centripetal and centrifugal velocities equal in magnitude and opposite in direction, which, interacting with each other, ensure the kinematic condition of uniform rotation of the material point relative to the stationary center of the circle, the radius of which is much greater than the geometric dimensions of the mother point are the additional parameters, which, together with the generally accepted kinematic parameters of the rotation of a material point determine the necessary and sufficient kinematic conditions for uniform rotation of a material point around a fixed axis.*
Keywords: *material point, uniform rotation; tangential speed; centrifugal acceleration; centripetal speed; centrifugal velocity, centrifugal speed, vector product.*

The issue of uniform free rotation[1] of a material point[2] around a circle relative to a fixed center displaced relative to a material point is the subject of a special study of one of the sections of kinematics and is considered to be thoroughly studied [1–3]. Among the main characteristics of this process are the uniform tangential velocity of a point in the direction of a uniform angular velocity of rotation of a material point along a circle and a non-zero normal acceleration of a material point directed in the direction towards the center of the circle [1–3]. At the same time, the radius of the circle remains constant and, thus, one of the main provisions of kinematics is violated, when in the direction of the constantly acting acceleration of a free material point, there is no speed and movement of the material point in the direction of this acceleration.

The aim of the work is to develop a method for vector analysis of the kinematic parameters of the uniform rotation of a material point around a fixed axis using the main kinematic parameters of a uniform rotation of a material point around a circle, such as normal centripetal acceleration and uniform tangential velocity which provides the relationship between the fundamental kinematic parameters (acceleration, movement speed) for a moving material point.

Let us consider (Fig. 1) the process of rotation of a material point (m) with a radius r_m with a uniform angular velocity $\omega = \text{const}$ in a horizontal plane along a circle with a radius $r >> r_m$ relative to a fixed vertical axis $0Z$ [1–3].

The value (module) of the uniform tangential velocity $v^\tau = v^\tau(t) = \left|\bar{v}^\tau(t)\right| = \left|\bar{v}^\tau\right| = \text{const}$ directed tangentially to the circle (perpendicular to the radius vector $\bar{r}$) in the direction of the angular velocity $\omega = \text{const}$ will be

$$v^\tau = \frac{L}{T} = \frac{2\pi r}{T} = \frac{2\pi r}{2\pi}\omega = \omega r = \left|\bar{\omega}\times\bar{r}\right| = \left|\bar{v}^\tau\right| \tag{1}$$

[1] Free random rotation of a material point (rigid body) represents a curvilinear motion of a material point in which the resultant moment of external forces relative to the moving center of mass of a material point (rigid body) is equal to zero [4].
[2] Under the term of material point in this article we will understand an absolutely rigid body, the center of mass of which coincides with a geometric point, connected with the position of the center of the ball (circle) with radius which is neglected when solving a specific problem of classical mechanics [5], [6].

where $L = 2\pi r$ is the length of circle, limited by the central angle 2π, along which moves m during one revolution; $T = \dfrac{2\pi}{\omega}$ - circumferential period of uniform rotation around the circle; $\overline{\omega}$ - vector of the angular velocity of rotation of the radius vector (not shown in fig. 1).

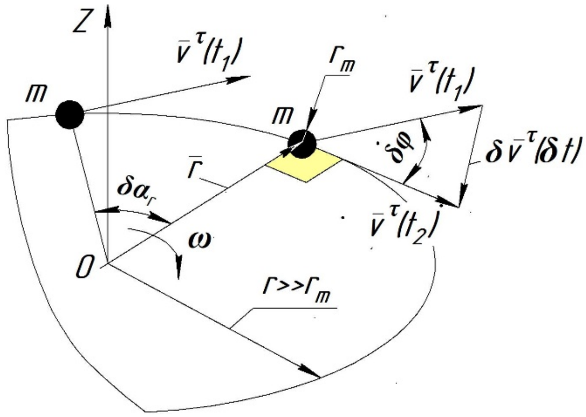

Fig.1.

When the radius of the vector $\bar{r}$ rotates by a small angle $\delta\alpha_r$, the tangential velocity vector $\bar{v}^{\tau}(t_1)$ will rotate by an angle over a period of time $\delta t = t_2 - t_1$

$$\delta\varphi = \delta\alpha_r = \omega\delta t \tag{2}$$

and will represent the tangential velocity vector $\bar{v}^{\tau}(t_2)$.

The segment between the vectors $\bar{v}^{\tau}(t_2)$ and $\bar{v}^{\tau}(t_1)$ in fig. 1 represents the relative rotation vector $\delta\bar{v}^{\tau}(\delta t)$ of the tangential velocity vector

$$\delta\bar{v}^{\tau}(\delta t) = \bar{v}^{\tau}(t_2) - \bar{v}^{\tau}(t_1) \tag{3}$$

which at $\delta\varphi = \delta\alpha_r << 2\pi$ ($\delta t << T$) is directed along the radius vector $\bar{r}$ to the center of the circle and will represent a small change in the tangential velocity vector

$$\delta v^{\tau} = \left|\delta\bar{v}^{\tau}(\delta t)\right| \approx v^{\tau}\delta\varphi = v^{\tau}\omega\delta t = \omega^2 r\delta t = \text{const} \tag{4}$$

Dividing δv^{τ} by the time interval δt, we obtain the modulus of normal centripetal acceleration $\bar{a}^n_{cp}$, directed along the normal to the center of the circle O [1–3].

$$\left|\bar{a}^n_{cp}\right| = \bar{a}^n_{cp} = \frac{\delta v^{\tau}}{\delta t} = \frac{\omega^2 r\delta t_i}{\delta t} = \omega^2 r \tag{5}$$

The calculation scheme for the rotation of a material point, shown in fig. 1. will correspond the vector design seme of this movement presented in fig. 2, *a*, where the direction of the angular velocity vector $\overline{\omega}$ in the direction opposite to the direction of the axis *0Z* is determined in the adopted calculation scheme in fig. 1 by the direction of rotation of the material point (m) along the circle clockwise [1–3].

Let us perform in the design vector scheme (Fig. 1) the following transformations (Fig. 2, *b*). We restore in the current position of mass m on a circle with a radius r the angular velocity vector $\overline{\omega}_m = \overline{\omega}$ and "balance" it with the vector $-\overline{\omega}_m$. By such a transformation, we do not violate the conditions m of rotation relative to the axis *0Z* presented in Fig. 1 and fig. 2, *a*. At the same time, in Fig. 2, *b*, two planes ($\mathrm{T}_{\omega m}$ and $\mathrm{T}_{-\omega m}$) can be distinguished in the form of rectangles, the sides of which, respectively, are vectors:

- $\overline{\omega}_m$ and $\overline{v}^\tau$ for plane $T_{\omega m}$ (painted in pink);
- $-\overline{\omega}$ and $\overline{v}^\tau$ for $T_{-\omega m}$ (painted in lettuce).

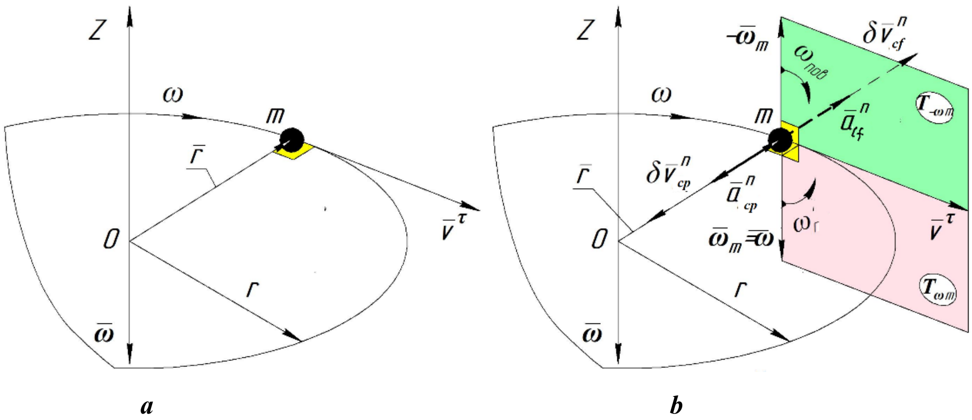

Fig. 2.

The product of vectors $\overline{\omega}$ and $\overline{v}^\tau$ in the plane $T_{\omega m}$ will determine the acceleration vector $\overline{a}_{cp}^n$. Its direction along the normal line to the plane $T_{\omega m}$ and along the radius-vector $\overline{r}$ towards the axis of rotation $0Z$ is determined using the “gimlet” rule when turning (ω_r) the vector $\overline{\omega}$ towards the vector $\overline{v}^\tau$ in the plane [1–3]

$$\left|\overline{a}_{\partial ц}^n\right| = \left|\overline{\omega}\times\overline{v}^\tau\right| = \omega v^\tau \sin(\pi/2) = \omega v^\tau = \omega^2 r \tag{6}$$

Note that the vector equation (5), which uniquely determines the magnitude and direction of the centripetal acceleration vector $\overline{a}_{cp}^n$, corresponds to the definition of the same vector $\overline{a}_{cp}^n$ when solving equations (2)-(4), which follow from the analysis of the calculation scheme shown in fig. 1.

Product of vectors $\overline{\omega}^{in}$ and $\overline{v}^\tau$ in the plane $T_{-\omega m}$ (fig. 2, *b*) is determined by the centrifugal acceleration vector $\overline{a}_{cf}^n$. Its direction is normal in the plane $T_{-\omega m}$ along the radius-vector $\overline{r}$ in the direction opposite to the axis of rotation OZ is defined using gimlet rule when turning the (ω_r) vector $-\overline{\omega}_m$ to the side of $\overline{v}^\tau$ in the plane $T_{-\omega m}$.

$$\overline{a}_{cf}^n = \left|-\overline{\omega}\times\overline{v}^\tau\right| = \omega v^\tau \sin(\pi/2) = \omega v^\tau = \omega^2 r \tag{7}$$

From (5) and (6) it follows

$$\overline{a}_{cf}^n = -\overline{a}_{cp}^n \tag{8}$$

From (5) taking into account the mutual direction of the vectors $\overline{a}_{cp}^n$ and $\overline{a}_{cf}^n$ we obtain the corresponding values of the centripetal velocity vectors $\delta\overline{v}_{cp}^n$ and centrifugal speed $\delta\overline{v}_{cf}^n$ rotation of a material point (m) around a circle (fig. 2, *b*):

$$\delta\overline{v}_{cp}^n = \overline{a}_{cp}^n \cdot \delta t \tag{9}$$

$$\delta\overline{v}_{cp}^n = \overline{a}_{cp}^n \cdot \delta t = -\overline{a}_{cf}^n \cdot \delta t \tag{10}$$

From (9) and (10) it follows

$$\delta\bar{v}_{cp}^{n} = -\delta\bar{v}_{cf}^{n} \tag{11}$$

Equations (8) and (11) determine the necessary and sufficient kinematic conditions for the steady free uniform rotation of a material point around a fixed vertical axis.

Conclusions. On the basis of the performed vector analysis of the free uniform rotation of a material point around a fixed vertical axis, additional kinematic parameters in the form of centrifugal acceleration and centrifugal speed are established, which, together with traditional kinematic parameters: tangential speed and centrifugal acceleration, provide the necessary and sufficient conditions for the uniform movement of a material point around a circle constant radius.

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До питання векторного аналізу рівномірного обертання матеріальної точки навколо нерухомої осі

І.В. Добров

Анотація. *На підставі проведеного векторного аналізу процесу рівномірного обертання матеріальної точки навколо нерухомої осі встановлено, що загальноприйняті кінематичні параметри обертання матеріальної точки по колу у вигляді тангенціальної швидкості та доцентрового прискорення є необхідними, але недостатніми для повної характеристики кінематики процесу обертання матеріальної точки навколо нерухомої осі. Додаткові параметри, які разом із загальноприйнятими кінематичними параметрами обертання матеріальної точки по колу, визначають необхідні і достатні кінематичні умови для рівномірного обертання матеріальної точки навколо нерухомої осі: відцентрове прискорення, що дорівнює за величиною і протилежне за напрямом доцентрового прискорення; рівні за величиною і протилежні за напрямком доцентрова і відцентрова швидкості, які взаємодіючі між собою забезпечують кінематичну умову рівномірного обертання матеріальної точки щодо нерухомого центру кола, радіус якого набагато перевищує геометричні розміри матеріальної точки.*

Ключові слова: *матеріальна точка, рівномірне обертання; тангенційна швидкість; доцентрове прискорення; відцентрове прискорення; доцентрова швидкість; відцентрова швидкість, векторний добуток.*